

GENERATORS FOR THE COHOMOLOGY RING OF HILBERT SCHEMES OF POINTS ON SURFACES

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ABSTRACT. Using the methods developed in [LQW], we obtain a second set of generators for the cohomology ring of the Hilbert scheme of points on an arbitrary smooth projective surface X over the field of complex numbers. These generators have clear and simple geometric as well as algebraic descriptions.

1. Introduction

In recent years, there has been a surge of research interest in Hilbert schemes $X^{[n]}$ of points on surfaces X largely due to the work of Göttsche [Got], Nakajima [Na1], Grojnowski [Gro], and Vafa and Witten [VW]. Earlier, Ellingsrud and Strømme [ES1] calculated the Betti numbers of the Hilbert schemes of points on the projective plane \mathbb{P}^2 , the affine plane \mathbb{C}^2 , and rational ruled surfaces. Subsequently, Göttsche [Got] determined the Betti numbers of $X^{[n]}$ for any surfaces X by finding a beautiful formula for the generating function of the Betti numbers of $X^{[n]}$. Göttsche's results suggested that one should study Hilbert schemes $X^{[n]}$ for all n altogether rather than study each $X^{[n]}$ individually. This idea is also echoed in Vafa and Witten's work [V-W] which stated that the generating function of the Euler numbers of $X^{[n]}$ is the partition function of some physical theory. Later on, Nakajima [Na1] constructed a Heisenberg algebra action on the direct sum of cohomology rings of $X^{[n]}$ over all n . Similar results were obtained by Grojnowski [Gro].

The cohomology ring structure of Hilbert schemes $X^{[n]}$ is a more subtle issue than the Betti numbers, and it has also been studied extensively. Many questions in enumerative geometry can be interpreted as questions in cup product in the cohomology ring of $X^{[n]}$ (see [EL]). Ellingsrud and Strømme in [ES2] studied the ring structure of the cohomology ring of $X^{[n]}$ when X is the projective plane \mathbb{P}^2 and the affine plane \mathbb{C}^2 . They found a set of generators of the cohomology ring structure on $H^*(X^{[n]})$ via the Chern classes of the tautological rank- n vector bundles coming from the universal subscheme. This result was extended by Beauville [Bea] to other rational surfaces and ruled surfaces. Very recently, Markman [Mar] further extended the method to K3 surfaces. This method, in the context of moduli spaces \mathfrak{M} of stable sheaves, basically says that if one can express the class of the diagonal in $\mathfrak{M} \times \mathfrak{M}$ in terms of the Chern classes of a universal sheaf \mathcal{E} on $\mathfrak{M} \times X$, then one can show that the Künneth components of Chern classes of \mathcal{E} provide a set of ring generators. Fantechi and Göttsche in [FG] also found a set of generators for the cohomology ring of $X^{[3]}$ for any surface X . There is a totally different approach initiated by Lehn [Leh] where the Heisenberg algebra construction mentioned earlier

is used in an essential way. In particular, Lehn was able to describe the cohomology ring of $(\mathbb{C}^2)^{[n]}$ in terms of certain explicit differential operators. There are also different viewpoints, as first indicated in the work of I. Frenkel and the third author [FW], of relating the cohomology rings of Hilbert schemes for the affine plane \mathbb{C}^2 to the convolution product on symmetric groups (see [LS1, Vas]).

In our paper [LQW], a set of $(n \cdot \dim H^*(X))$ generators for the cohomology ring of $X^{[n]}$ for any smooth projective surface X was found. These generators, denoted by $G_i(\gamma, n)$ where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$, were defined by using essentially the Chern classes of the universal subscheme (see Definition 2.8 (vi) for details). They coincide with the generators found by Ellingsrud and Strømme in the case of \mathbb{P}^2 . Note that although the statement on this set of generators has nothing to do with Heisenberg algebras etc, the machinery which we built up in its proof is deeply rooted in the theory of vertex algebras and the work of Nakajima, Grojnowski and Lehn. We took the viewpoint effectively that the cup products with $G_i(\gamma, n)$ for all n associated to a fixed γ should be treated as a single operator acting on the direct sum of cohomology rings of $X^{[n]}$ over all n .

The geometric interpretation of the ring generators in terms of the universal subscheme is not always clear however. In this paper, we present a new set of $(n \cdot \dim H^*(X))$ generators for the cohomology rings of Hilbert schemes, which has a simple geometric interpretation. This new set of generators also affords a nice algebraic interpretation in terms of the Heisenberg algebra operators. To be precise, let $|0\rangle$ be the element 1 of $H^0(X^{[0]}) = \mathbb{Q}$. Denote by $q_j(\gamma)$ the Heisenberg algebra operators, where $j \in \mathbb{Z}$ and $\gamma \in H^*(X)$ (see Definition 2.8 (ii)). For $0 \leq i < n$ and $\gamma \in H^*(X)$, define a cohomology class $B_i(\gamma, n) \in H^*(X^{[n]})$ by putting

$$B_i(\gamma, n) = \frac{1}{(n-i-1)!} \cdot q_{i+1}(\gamma) q_1(1_X)^{n-i-1} |0\rangle.$$

Note that these are the simplest cohomology classes in $H^*(X^{[n]})$ both in geometric terms and Heisenberg algebraic terms. Moreover, $B_0(1_X, n) = n \cdot 1_{X^{[n]}}$. In addition, if either $i > 0$ or $\gamma \in H^s(X)$ with $s > 0$, then one can easily show that the Poincaré dual of $B_i(\gamma, n)$ is the homology class represented by the closed subset:

$$\{ \xi \in X^{[n]} \mid \exists x \in \Gamma \text{ with } \ell(\xi_x) \geq i+1 \}$$

where Γ is a homology cycle of X representing the Poincaré dual of γ , and ξ_x is the component of ξ such that ξ_x is supported at x .

The following is our main result in this paper.

Theorem 1.1. *Let X be a smooth projective surface over the field of complex numbers. For $n \geq 1$, the cohomology ring $\mathbb{H}_n = H^*(X^{[n]})$ is generated by the classes $B_i(\gamma, n)$ where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$.*

This theorem is proved in section 3 after we review some definitions and results from [Na2, Leh, LQW] in Section 2. The main idea is that by exploring the results established in [LQW], we are able to determine certain relations among the cohomology classes $G_i(\gamma, n)$ introduced in [LQW] and $B_i(\gamma, n)$ introduced here. More precisely, we show that $B_i(\gamma, n)$ coincides with $G_i(\gamma, n)$ for $i = 0, 1$. In addition, for $2 \leq i < n$, $B_i(\gamma, n)$ is equal to $(-1)^i(i+1)! \cdot G_i(\gamma, n)$ plus a finite sum of products of the form

$$G_{m_1}(\gamma_1, n) \cdot \dots \cdot G_{m_t}(\gamma_t, n) \in \mathbb{H}_n$$

where $m_1, \dots, m_t \geq 0$ with $m_1 + \dots + m_t < i$, and $\gamma_1, \dots, \gamma_t \in H^*(X)$. We remark that in the process of deriving the relations among $G_i(\gamma, n)$ and $B_i(\gamma, n)$, we actually find a new proof to our theorem in [LQW] which states that the classes $G_i(\gamma, n)$, where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$, generate the cohomology ring of $X^{[n]}$. Then Theorem 1.1 is derived by using this theorem and the relations between the cohomology classes $G_i(\gamma, n)$ and $B_i(\gamma, n)$.

Conventions: Throughout the paper, all cohomology rings are in \mathbb{Q} -coefficients. The cup product between two cohomology classes α and β is denoted by $\alpha \cdot \beta$ or simply by $\alpha\beta$. For a continuous map $p : Y_1 \rightarrow Y_2$ between two smooth compact manifolds and for $\alpha_1 \in H^*(Y_1)$, the push-forward $p_*(\alpha_1)$ is defined by

$$p_*(\alpha_1) = \text{PD}^{-1} p_*(\text{PD}(\alpha_1))$$

where PD stands for the Poincaré duality. Unless otherwise specified, we make no distinction between an algebraic cycle and its corresponding cohomology class so that intersections among algebraic cycles correspond to cup products among the corresponding cohomology classes. For instance, for two algebraic cycles $[a]$ and $[b]$ on a smooth projective variety Y , it is understood that $[a] \cdot [b] \in H^*(Y)$.

2. Results from [Na2, Leh, LQW]

In this section, we shall fix some notations, and recall some results from [Na2, Leh, LQW]. For convenience, we also review certain basic facts for the Hilbert scheme of points in a smooth projective surface.

Let X be a smooth projective surface over \mathbb{C} , and $X^{[n]}$ be the Hilbert scheme of points in X . An element in the Hilbert scheme $X^{[n]}$ is represented by a length- n 0-dimensional closed subscheme ξ of X , which sometimes is called a length- n 0-cycle. For $\xi \in X^{[n]}$, let I_ξ and \mathcal{O}_ξ be the corresponding sheaf of ideals and structure sheaf respectively. For a point $x \in X$, let ξ_x be the component of ξ supported at x and $I_{\xi, x} \subset \mathcal{O}_{X, x}$ be the stalk of I_ξ at x . It is known from [Fog] that $X^{[n]}$ is smooth. In $X^{[n]} \times X$, we have the universal codimension-2 subscheme:

$$\mathcal{Z}_n = \{(\xi, x) \in X^{[n]} \times X \mid x \in \text{Supp}(\xi)\} \subset X^{[n]} \times X. \quad (2.1)$$

Also, we let $X^n = \underbrace{X \times \dots \times X}_{n \text{ times}}$ be the n -th Cartesian product, and

$$X^{[n_1], \dots, [n_k]} = X^{[n_1]} \times \dots \times X^{[n_k]}. \quad (2.2)$$

Definition 2.3. (i) Let $\mathbb{H} = \bigoplus_{n, i \geq 0} \mathbb{H}^{n, i}$ denote the double-graded vector space with components $\mathbb{H}^{n, i} \stackrel{\text{def}}{=} H^i(X^{[n]})$, and $\mathbb{H}_n \stackrel{\text{def}}{=} H^*(X^{[n]}) \stackrel{\text{def}}{=} \bigoplus_{i=0}^{4n} H^i(X^{[n]})$. The element 1 in $H^0(X^{[0]}) = \mathbb{Q}$ is called the *vacuum vector* and denoted by $|0\rangle$;

(ii) A linear operator $\mathfrak{f} \in \text{End}(\mathbb{H})$ is *homogeneous of bidegree* (ℓ, m) if

$$\mathfrak{f}(\mathbb{H}^{n, i}) \subset \mathbb{H}^{n+\ell, i+m}. \quad (2.4)$$

Furthermore, $\mathfrak{f} \in \text{End}(\mathbb{H})$ is *even* (resp., *odd*) if m is even (resp., odd).

(iii) For two homogeneous linear operators \mathfrak{f} and $\mathfrak{g} \in \text{End}(\mathbb{H})$ of bidegrees (ℓ, m) and (ℓ_1, m_1) respectively, define the *Lie superalgebra bracket* $[\mathfrak{f}, \mathfrak{g}]$ by

$$[\mathfrak{f}, \mathfrak{g}] = \mathfrak{f}\mathfrak{g} - (-1)^{mm_1} \mathfrak{g}\mathfrak{f}. \quad (2.5)$$

(iv) Let $\mathfrak{A}(\alpha), \mathfrak{B}(\beta) \in \text{End}(\mathbb{H})$ be two series of operators depending linearly on $\alpha, \beta \in H^*(X)$. Then, the commutator $[\mathfrak{A}(\alpha), \mathfrak{B}(\beta)]$ satisfies *the transfer property* if $[\mathfrak{A}(\alpha), \mathfrak{B}(\beta)] = [\mathfrak{A}(1_X), \mathfrak{B}(\alpha\beta)] = [\mathfrak{A}(\alpha\beta), \mathfrak{B}(1_X)]$ for all α, β .

A non-degenerate super-symmetric bilinear form $(,)$ on \mathbb{H} is induced from the standard one on $\mathbb{H}_n = H^*(X^{[n]})$. For a homogeneous linear operator $\mathfrak{f} \in \text{End}(\mathbb{H})$ of bidegree (ℓ, m) , we can define its *adjoint* $\mathfrak{f}^\dagger \in \text{End}(\mathbb{H})$ by

$$(\mathfrak{f}(\alpha), \beta) = (-1)^{m \cdot |\alpha|} \cdot (\alpha, \mathfrak{f}^\dagger(\beta)) \quad (2.6)$$

where $|\alpha| = s$ for $\alpha \in H^s(X)$. Note that the bidegree of \mathfrak{f}^\dagger is $(-\ell, m - 4\ell)$. Also,

$$(\mathfrak{f}\mathfrak{g})^\dagger = (-1)^{mm_1} \cdot \mathfrak{g}^\dagger \mathfrak{f}^\dagger \quad \text{and} \quad [\mathfrak{f}, \mathfrak{g}]^\dagger = -[\mathfrak{f}^\dagger, \mathfrak{g}^\dagger] \quad (2.7)$$

where $\mathfrak{g} \in \text{End}(\mathbb{H})$ is another homogeneous linear operator of bidegree (ℓ_1, m_1) .

Next, we collect from [Na2, Leh, LQW] the definitions of the closed subset $Q^{[n+\ell, n]}$ in $X^{[n+\ell]} \times X \times X^{[n]}$, the Heisenberg generator \mathfrak{q}_n , the Virasoro generator \mathfrak{L}_n , the boundary operator \mathfrak{d} , the derivative \mathfrak{f}' of a linear operator $\mathfrak{f} \in \text{End}(\mathbb{H})$, and the operators $\mathfrak{G}_i(\alpha) \in \text{End}(\mathbb{H})$ for $i \geq 0$ and $\alpha \in H^*(X)$.

Definition 2.8. (i) For $n \geq 0$, define $Q^{[n, n]} = \emptyset$. For $n \geq 0$ and $\ell > 0$, define $Q^{[n+\ell, n]} \subset X^{[n+\ell]} \times X \times X^{[n]}$ to be the closed subset

$$\{(\xi, x, \eta) \in X^{[n+\ell]} \times X \times X^{[n]} \mid \xi \supset \eta \text{ and } \text{Supp}(I_\eta/I_\xi) = \{x\}\}; \quad (2.9)$$

(ii) For $n \in \mathbb{Z}$, define linear maps $\mathfrak{q}_n : H^*(X) \rightarrow \text{End}(\mathbb{H})$ as follows. When $n \geq 0$, the linear operator $\mathfrak{q}_n(\alpha) \in \text{End}(\mathbb{H})$ with $\alpha \in H^*(X)$ is defined by

$$\mathfrak{q}_n(\alpha)(a) = \tilde{p}_{1*}([Q^{[m+n, m]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2^* a) \quad (2.10)$$

for all $a \in \mathbb{H}_m = H^*(X^{[m]})$, where $[Q^{[m+n, m]}]$ is (the cohomology class corresponding to) the algebraic cycle associated to $Q^{[m+n, m]}$, and $\tilde{p}_1, \tilde{\rho}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively. When $n < 0$, define the operator $\mathfrak{q}_n(\alpha) \in \text{End}(\mathbb{H})$ with $\alpha \in H^*(X)$ by

$$\mathfrak{q}_n(\alpha) = (-1)^n \cdot \mathfrak{q}_{-n}(\alpha)^\dagger; \quad (2.11)$$

(iii) For $n \in \mathbb{Z}$, define linear maps $\mathfrak{L}_n : H^*(X) \rightarrow \text{End}(\mathbb{H})$ by putting

$$\mathfrak{L}_n = \begin{cases} \frac{1}{2} \cdot \sum_{m \in \mathbb{Z}} \mathfrak{q}_m \mathfrak{q}_{n-m} \tau_{2*}, & \text{if } n \neq 0 \\ \sum_{m > 0} \mathfrak{q}_m \mathfrak{q}_{-m} \tau_{2*}, & \text{if } n = 0 \end{cases} \quad (2.12)$$

where $\tau_{2*} : H^*(X) \rightarrow H^*(X^2)$ is the linear map induced by the diagonal embedding $\tau_2 : X \rightarrow X^2$, and the operator $\mathfrak{q}_m \mathfrak{q}_\ell \tau_{2*}(\alpha)$ stands for

$$\sum_j \mathfrak{q}_m(\alpha_{j,1}) \mathfrak{q}_\ell(\alpha_{j,2}) \quad (2.13)$$

when $\tau_{k*}\alpha = \sum_j \alpha_{j,1} \otimes \alpha_{j,2}$ via the Künneth decomposition of $H^*(X^2)$;

(iv) Define the linear operator $\mathfrak{d} \in \text{End}(\mathbb{H})$ by

$$\mathfrak{d} = \bigoplus_n c_1(p_{1*}\mathcal{O}_{\mathcal{Z}_n}) = \bigoplus_n (-[\partial X^{[n]}]/2) \quad (2.14)$$

where p_1 is the projection of $X^{[n]} \times X$ to $X^{[n]}$, $\partial X^{[n]}$ is the boundary of $X^{[n]}$ consisting of all $\xi \in X^{[n]}$ with $|\text{Supp}(\xi)| < n$, and the first Chern class $c_1(p_{1*}\mathcal{O}_{\mathcal{Z}_n})$ of the rank- n bundle $p_{1*}\mathcal{O}_{\mathcal{Z}_n}$ acts on $\mathbb{H}_n = H^*(X^{[n]})$ by the cup product.

(v) For a linear operator $\mathfrak{f} \in \text{End}(\mathbb{H})$, define its *derivative* \mathfrak{f}' by

$$\mathfrak{f}' \stackrel{\text{def}}{=} [\mathfrak{d}, \mathfrak{f}]. \quad (2.15)$$

The higher derivative $\mathfrak{f}^{(k)}$ of \mathfrak{f} is defined inductively by $\mathfrak{f}^{(k)} = [\mathfrak{d}, \mathfrak{f}^{(k-1)}]$.

(vi) For $\alpha \in H^*(X)$ and $n \geq 0$, let $G_i(\alpha, n)$ be the $H^{|\alpha|+2i}(X^{[n]})$ -component of

$$p_{1*}(\text{ch}(\mathcal{O}_{\mathcal{Z}_n}) \cdot p_2^* \text{td}(X) \cdot p_2^* \alpha) \in \mathbb{H}_n$$

where p_2 is the projection of $X^{[n]} \times X$ to X . For $i \geq 0$ and $\alpha \in H^*(X)$, define $\mathfrak{G}_i(\alpha) \in \text{End}(\mathbb{H})$ to be the operator which acts on the component $\mathbb{H}_n = H^*(X^{[n]})$ by the cup product by the cohomology class $G_i(\alpha, n)$.

Theorem 2.16. *Let K_X and $c_2(X)$ be the canonical divisor and the second Chern class of X respectively. Let $k \geq 0, n, m \in \mathbb{Z}$ and $\alpha, \beta \in H^*(X)$. Then,*

- (i) $[\mathfrak{q}_n(\alpha), \mathfrak{q}_m(\beta)] = n \cdot \delta_{n+m} \cdot \int_X (\alpha\beta) \cdot \text{Id}_{\mathbb{H}};$
- (ii) $[\mathfrak{L}_n(\alpha), \mathfrak{q}_m(\beta)] = -m \cdot \mathfrak{q}_{n+m}(\alpha\beta);$
- (iii) $[\mathfrak{L}_n(\alpha), \mathfrak{L}_m(\beta)] = (n-m) \cdot \mathfrak{L}_{n+m}(\alpha\beta) - \frac{n^3-n}{12} \cdot \delta_{n+m} \cdot \int_X (c_2(X)\alpha\beta) \cdot \text{Id}_{\mathbb{H}};$
- (iv) $\mathfrak{q}'_n(\alpha) = n \cdot \mathfrak{L}_n(\alpha) + \frac{n(|n|-1)}{2} \mathfrak{q}_n(K_X \alpha);$
- (v) $[\mathfrak{G}_k(\alpha), \mathfrak{q}_1(\beta)] = \frac{1}{k!} \cdot \mathfrak{q}_1^{(k)}(\alpha\beta);$
- (vi) $[\dots [\mathfrak{G}_k(\alpha), \mathfrak{q}_{n_1}(\alpha_1)], \dots], \mathfrak{q}_{n_{k+1}}(\alpha_{k+1})]$ is equal to

$$- \prod_{\ell=1}^{k+1} (-n_\ell) \cdot \mathfrak{q}_{n_1+\dots+n_{k+1}}(\alpha\alpha_1 \cdots \alpha_{k+1})$$

for all $n_1, \dots, n_{k+1} \in \mathbb{Z}$ with $\sum_{\ell=1}^{k+1} n_\ell \neq 0$ and all $\alpha_1, \dots, \alpha_{k+1} \in H^*(X)$.

We notice that Theorem 2.16 (i) was proved by Nakajima [Na1] subject to some universal nonzero constant, which was determined subsequently in [ES3]. The next three formulas in Theorem 2.16 were obtained by Lehn [Leh]. Theorem 2.16 (v) is the Lemma 5.8 in [LQW], which is a generalization of a remarkable theorem of Lehn ([Leh], Theorem 4.2). Theorem 2.16 (vi) follows from the Theorem 5.13 (ii) and Proposition 4.10 (i) in [LQW], and the proof of it uses the part (v) above. Also, as observed by Nakajima and Grojnowski in [Na1, Gro], \mathbb{H} is an irreducible representation of the Heisenberg algebra generated by the $\mathfrak{q}_i(\alpha)$'s with $|0\rangle \in H^0(X^{[0]})$ being the highest weight vector. So a linear basis of \mathbb{H} is given

by $\mathfrak{q}_{i_1}(\alpha_1)\mathfrak{q}_{i_2}(\alpha_2)\cdots\mathfrak{q}_{i_k}(\alpha_k)|0\rangle$ where $k \geq 0$, $i_1 \geq i_2 \geq \cdots \geq i_k > 0$, and each of $\alpha_1, \alpha_2, \dots, \alpha_k$ runs over a fixed linear basis of $H^*(X)$.

The following elementary but convenient result is the Lemma 5.26 in [LQW].

Lemma 2.17. *Fix a, b with $1 \leq a \leq b$. Let $\mathfrak{g} \in \text{End}(\mathbb{H})$ be of bidegree (ℓ, s) , and*

$$A = \mathfrak{q}_{m_1}(\beta_1) \cdots \mathfrak{q}_{m_b}(\beta_b)|0\rangle.$$

Then, $\mathfrak{g}(A)$ is equal to the sum of the following two terms:

$$\sum_{i=0}^{a-1} \sum_{\sigma_i} \pm \prod_{\ell \in \sigma_i^0} \mathfrak{q}_{m_\ell}(\beta_\ell) [\cdots [\mathfrak{g}, \mathfrak{q}_{m_{\sigma_i(1)}}(\beta_{\sigma_i(1)}), \cdots], \mathfrak{q}_{m_{\sigma_i(i)}}(\beta_{\sigma_i(i)})] |0\rangle \quad (2.18)$$

and

$$\begin{aligned} & \sum_{\sigma_a} (-1)^{\sum_{k=0}^{a-1} (s + \sum_{\ell=1}^k |\beta_{\sigma_a(\ell)}|) \sum_{\sigma_a(k) < j < \sigma_a(k+1)} |\beta_j|} \prod_{\ell \in \sigma_a^1} \mathfrak{q}_{m_\ell}(\beta_\ell) \cdot \\ & \cdot [\cdots [\mathfrak{g}, \mathfrak{q}_{m_{\sigma_a(1)}}(\beta_{\sigma_a(1)}), \cdots], \mathfrak{q}_{m_{\sigma_a(a)}}(\beta_{\sigma_a(a)})] \prod_{\ell \in \sigma_a^2} \mathfrak{q}_{m_\ell}(\beta_\ell) |0\rangle \end{aligned} \quad (2.19)$$

where for each fixed i with $0 \leq i \leq a$, σ_i runs over all the maps

$$\{1, \dots, i\} \rightarrow \{1, \dots, b\}$$

satisfying $\sigma_i(1) < \cdots < \sigma_i(i)$. Moreover, $\sigma_i^0 = \{\ell \mid 1 \leq \ell \leq b, \ell \neq \sigma_i(1), \dots, \sigma_i(i)\}$, $\sigma_a^1 = \{\ell \mid 1 \leq \ell < \sigma_a(a), \ell \neq \sigma_a(1), \dots, \sigma_a(a)\}$, and $\sigma_a^2 = \{\ell \mid \sigma_a(a) < \ell \leq b\}$.

Proof. Follows from moving all the commutators

$$[\cdots [\mathfrak{g}, \mathfrak{q}_{m_{\sigma_i(1)}}(\beta_{\sigma_i(1)}), \cdots], \mathfrak{q}_{m_{\sigma_i(i)}}(\beta_{\sigma_i(i)})]$$

with $0 \leq i \leq (a-1)$ to the right, and applying the fact that

$$\mathfrak{g}_1 \mathfrak{g}_2 = [\mathfrak{g}_1, \mathfrak{g}_2] + (-1)^{m_1 m_2} \mathfrak{g}_2 \mathfrak{g}_1$$

for $\mathfrak{g}_1, \mathfrak{g}_2 \in \text{End}(\mathbb{H})$ of bidegrees $(\ell_1, m_1), (\ell_2, m_2)$ respectively. \square

3. A second set of generators for the cohomology ring

In this section, we shall prove that the $(n \cdot \dim H^*(X))$ cohomology classes $B_i(\gamma, n)$ in Theorem 1.1 generate the cohomology ring $\mathbb{H}_n = H^*(X^{[n]})$. Moreover, we obtain an alternative proof to the Theorem 5.30 in [LQW] which has been proved there by a different method. Finally, letting $\mathfrak{B}_i(\gamma) \in \text{End}(\mathbb{H})$ be the operator defined in (3.3), we show that the commutator $[\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)]$ satisfies the transfer property (see Definition 2.3 (iv) for the definition of the transfer property).

Definition 3.1. (i) For $0 \leq i < n$ and $\gamma \in H^*(X)$, define

$$B_i(\gamma, n) = \frac{1}{(n-i-1)!} \cdot \mathfrak{q}_{i+1}(\gamma) \mathfrak{q}_1(1_X)^{n-i-1} |0\rangle \in \mathbb{H}_n; \quad (3.2)$$

(ii) For $i \geq 0$ and $\gamma \in H^*(X)$, define the operator $\mathfrak{B}_i(\gamma) \in \text{End}(\mathbb{H})$ by

$$\mathfrak{B}_i(\gamma) = \bigoplus_{n \geq 0} B_i(\gamma, n) \quad (3.3)$$

where $B_i(\gamma, n)$ acts on the component $\mathbb{H}_n = H^*(X^{[n]})$ by the cup product.

Notice that $\mathfrak{B}_i(\gamma) \in \text{End}(\mathbb{H})$ is homogeneous of bidegree $(0, |\gamma| + 2i)$. Also,

$$\mathfrak{B}_i(\gamma)' = 0 \quad \text{and} \quad \mathfrak{B}_i(\gamma)^\dagger = \mathfrak{B}_i(\gamma). \quad (3.4)$$

Our goal is to show that the cohomology ring $\mathbb{H}_n = H^*(X^{[n]})$ is generated by the $(n \cdot \dim H^*(X))$ classes $B_i(\gamma, n)$ where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$. We shall start with a technical lemma which allows us to present an alternative proof to the Theorem 5.30 in [LQW] (see Theorem 3.19 below). Recall the cohomology class $G_i(\gamma, n)$ defined in Definition 2.8 (vi).

Lemma 3.5. *For $1 \leq i \leq n$, let $\mathbb{H}'_{n,i}$ be the linear span of all the classes*

$$G_{m_1}(\gamma_1, n) \cdot \dots \cdot G_{m_t}(\gamma_t, n) \in \mathbb{H}_n \quad (3.6)$$

where $m_1, \dots, m_t \geq 0$ with $m_1 + \dots + m_t < i$, and $\gamma_1, \dots, \gamma_t \in H^*(X)$. Then,

$$\mathfrak{q}_1(1_X)^{n-i} \mathfrak{q}_{n_1}(\alpha_1) \cdots \mathfrak{q}_{n_k}(\alpha_k) |0\rangle \in \mathbb{H}'_{n,i} \quad (3.7)$$

for all positive integers n_1, \dots, n_k with $\sum_{\ell=1}^k n_\ell = i$ and all $\alpha_1, \dots, \alpha_k \in H^*(X)$.

Proof. We shall use induction on i . First of all, assume $i = 1$. Then, $k = 1$ and $n_1 = 1$. By Theorem 2.16 (v), $[\mathfrak{G}_0(\alpha_1), \mathfrak{q}_1(1_X)] = \mathfrak{q}_1(\alpha_1)$. Thus, we have

$$\begin{aligned} (n-1)! \cdot G_0(\alpha_1, n) &= (n-1)! \cdot \mathfrak{G}_0(\alpha_1) 1_{X^{[n]}} \\ &= \frac{1}{n} \cdot \mathfrak{G}_0(\alpha_1) \mathfrak{q}_1(1_X)^n |0\rangle = \mathfrak{q}_1(1_X)^{n-1} \mathfrak{q}_1(\alpha_1) |0\rangle \end{aligned} \quad (3.8)$$

noting that $1_{X^{[n]}} = \frac{1}{n!} \cdot \mathfrak{q}_1(1_X)^n |0\rangle$ and $\mathfrak{G}_0(\alpha_1) |0\rangle = 0$. So (3.7) is true when $i = 1$.

Next, fixing an integer i_0 satisfying $1 \leq i_0 < n$, we assume that (3.7) holds for all the integers i with $1 \leq i \leq i_0$. We shall prove that (3.7) holds as well for $i = (i_0 + 1)$. In other words, we shall verify that

$$\mathfrak{q}_1(1_X)^{n-(i_0+1)} \mathfrak{q}_{n_1}(\alpha_1) \cdots \mathfrak{q}_{n_k}(\alpha_k) |0\rangle \in \mathbb{H}'_{n, i_0+1} \quad (3.9)$$

for all positive integers n_1, \dots, n_k with $\sum_{\ell=1}^k n_\ell = i_0 + 1$ and all $\alpha_1, \dots, \alpha_k \in H^*(X)$.

Let us explain how our induction works before we get into the details. We shall take a suitably chosen element A which lies in \mathbb{H}'_{n, i_0} , and consider $\mathfrak{G}_m(\alpha)(A) = G_m(\alpha, n) \cdot A$ which lies in \mathbb{H}'_{n, i_0+1} for some suitably chosen $m \geq 0$ and $\alpha \in H^*(X)$. By applying Lemma 2.17 to the situation at hand, we shall observe that the summation (2.18) in $\mathfrak{G}_m(\alpha)(A)$ is in \mathbb{H}'_{n, i_0} , and all the terms in (2.19) except those which coincide with (3.9) are also in \mathbb{H}'_{n, i_0} . Since $\mathbb{H}'_{n, i_0} \subset \mathbb{H}'_{n, i_0+1}$, (3.9) follows.

Now let us be more precise. Since $\mathbb{H}'_{n,i_0} \subset \mathbb{H}'_{n,i_0+1}$, by induction hypothesis, we may assume that $\mathbf{q}_{n_\ell}(\alpha_\ell) \neq \mathbf{q}_1(1_X)$ for all the integers ℓ with $1 \leq \ell \leq k$. Put

$$\begin{aligned} A &= \mathbf{q}_1(1_X)^{n-(i_0+1-n_1)} \mathbf{q}_{n_2}(\alpha_2) \cdots \mathbf{q}_{n_k}(\alpha_k) |0\rangle \\ &\stackrel{\text{def}}{=} \mathbf{q}_{m_1}(\beta_1) \cdots \mathbf{q}_{m_b}(\beta_b) |0\rangle \end{aligned} \quad (3.10)$$

where we have $b = (n - (i_0 + 1 - n_1)) + (k - 1)$, and

$$\mathbf{q}_{m_\ell}(\beta_\ell) = \mathbf{q}_1(1_X) \quad \text{for } 1 \leq \ell \leq n - (i_0 + 1 - n_1) \quad (3.11)$$

$$\mathbf{q}_{m_\ell}(\beta_\ell) \neq \mathbf{q}_1(1_X) \quad \text{for } n - (i_0 + 1 - n_1) < \ell \leq b. \quad (3.12)$$

Consider $G_{n_1-1}(\alpha_1, n) \cdot A$. Note that $0 \leq (i_0 + 1 - n_1) \leq i_0$. If $(i_0 + 1 - n_1) = 0$, then $A = \mathbf{q}_1(1_X)^n |0\rangle = (n-1)! \cdot G_0(1_X, n) \in \mathbb{H}'_{n,1}$ by (3.8); so

$$G_{n_1-1}(\alpha_1, n) \cdot A \in G_{n_1-1}(\alpha_1, n) \cdot \mathbb{H}'_{n,1} \subset \mathbb{H}'_{n,n_1} = \mathbb{H}'_{n,i_0+1}. \quad (3.13)$$

If $(i_0 + 1 - n_1) > 0$, then by induction hypothesis, $A \in \mathbb{H}'_{n,i_0+1-n_1}$; so

$$G_{n_1-1}(\alpha_1, n) \cdot A \in G_{n_1-1}(\alpha_1, n) \cdot \mathbb{H}'_{n,i_0+1-n_1} \subset \mathbb{H}'_{n,i_0} \subset \mathbb{H}'_{n,i_0+1}. \quad (3.14)$$

In summary, we have showed that $G_{n_1-1}(\alpha_1, n) \cdot A \in \mathbb{H}'_{n,i_0+1}$. Thus,

$$\mathfrak{G}_{n_1-1}(\alpha_1)(A) = G_{n_1-1}(\alpha_1, n) \cdot A \in \mathbb{H}'_{n,i_0+1}. \quad (3.15)$$

Applying Lemma 2.17 to $a = n_1$ and $\mathfrak{g} = \mathfrak{G}_{n_1-1}(\alpha_1) = \mathfrak{G}_{a-1}(\alpha_1)$, we see that the class $\mathfrak{G}_{a-1}(\alpha_1)(A)$ consists of two parts (2.18) and (2.19). By (3.11), the number of $\mathbf{q}_1(1_X)$'s in every nonvanishing term of (2.18) is at least

$$(n - (i_0 + 1 - n_1)) - (a - 1) = n - i_0. \quad (3.16)$$

So by induction hypothesis, (2.18) is contained in \mathbb{H}'_{n,i_0} . Let $N(\sigma_a)$ be the number of $\mathbf{q}_1(1_X)$'s in a nonvanishing term in (2.19) corresponding to σ_a . By (3.11) again,

$$N(\sigma_a) \geq (n - (i_0 + 1 - n_1)) - a = n - (i_0 + 1). \quad (3.17)$$

If $N(\sigma_a) > n - (i_0 + 1)$, then by induction, this nonvanishing term in (2.19) corresponding to σ_a is contained in \mathbb{H}'_{n,i_0} . Moreover, we see from (3.12) that

$$N(\sigma_a) = n - (i_0 + 1)$$

if and only if $\sigma_a(1) < \cdots < \sigma_a(a) \leq n - (i_0 + 1 - n_1)$. So this nonvanishing term

$$\begin{aligned} &(-1)^{\sum_{k=0}^{a-1} (s + \sum_{\ell=1}^k |\beta_{\sigma_a(\ell)}|) \sum_{\sigma_a(k) < j < \sigma_a(k+1)} |\beta_j|} \prod_{\ell \in \sigma_a^1} \mathbf{q}_{m_\ell}(\beta_\ell) \cdot \\ &\cdot [\cdots [\mathfrak{G}_{a-1}(\alpha_1), \mathbf{q}_{m_{\sigma_a(1)}}(\beta_{\sigma_a(1)}), \cdots], \mathbf{q}_{m_{\sigma_a(a)}}(\beta_{\sigma_a(a)})] \prod_{\ell \in \sigma_a^2} \mathbf{q}_{m_\ell}(\beta_\ell) |0\rangle \end{aligned}$$

in (2.19) corresponding to σ_a can be simplified to

$$\mathfrak{q}_1(1_X)^{|\sigma_a^1|} [\cdots [\mathfrak{G}_{a-1}(\alpha_1), \underbrace{\mathfrak{q}_1(1_X), \cdots}_{a \text{ times}}, \mathfrak{q}_1(1_X)] \prod_{\ell \in \sigma_a^2} \mathfrak{q}_{m_\ell}(\beta_\ell) |0\rangle.$$

By Theorem 2.16 (vi), the above term can be further simplified to

$$\begin{aligned} & \mathfrak{q}_1(1_X)^{|\sigma_a^1|} (-1)^{a+1} \mathfrak{q}_a(\alpha_1) \prod_{\ell \in \sigma_a^2} \mathfrak{q}_{m_\ell}(\beta_\ell) |0\rangle \\ &= (-1)^{n_1+1} \cdot \mathfrak{q}_1(1_X)^{n-(i_0+1)} \mathfrak{q}_{n_1}(\alpha_1) \cdots \mathfrak{q}_{n_k}(\alpha_k) |0\rangle. \end{aligned}$$

Now there are $\binom{n-(i_0+1-n_1)}{a} = \binom{n-(i_0+1-n_1)}{n_1}$ such terms in (2.19). Therefore,

$$\begin{aligned} & \mathfrak{G}_{n_1-1}(\alpha_1)(A) = \mathfrak{G}_{a-1}(\alpha_1)(A) \\ & \equiv \binom{n-(i_0+1)+n_1}{n_1} \cdot (-1)^{n_1+1} \\ & \quad \cdot \mathfrak{q}_1(1_X)^{n-(i_0+1)} \mathfrak{q}_{n_1}(\alpha_1) \cdots \mathfrak{q}_{n_k}(\alpha_k) |0\rangle \end{aligned} \quad (3.18)$$

modulo \mathbb{H}'_{n,i_0} . Since $\mathbb{H}'_{n,i_0} \subset \mathbb{H}'_{n,i_0+1}$, (3.9) follows from (3.15) and (3.18). \square

Theorem 3.19. *For $n \geq 1$, the cohomology ring $\mathbb{H}_n = H^*(X^{[n]})$ is generated by the classes $G_i(\gamma, n)$ where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$.*

Proof. Follows immediately from (3.7) by taking $i = n$. We remark that this is the Theorem 5.30 in [LQW], and has been proved there by a different method. \square

The next lemma provides a relation between the classes $B_i(\gamma, n)$ and $G_i(\gamma, n)$.

Lemma 3.20. (i) *For $\gamma \in H^*(X)$, we have*

$$B_0(\gamma, n) = G_0(\gamma, n) \quad \text{and} \quad B_1(\gamma, n) = -2G_1(\gamma, n);$$

(ii) *For $2 \leq i < n$ and $\gamma \in H^*(X)$, we have*

$$B_i(\gamma, n) \equiv (-1)^i (i+1)! \cdot G_i(\gamma, n) \pmod{\mathbb{H}'_{n,i}}.$$

Proof. (i) Recall from (3.2) that for $0 \leq i < n$, we have

$$(n-i-1)! \cdot B_i(\gamma, n) = \mathfrak{q}_{i+1}(\gamma) \mathfrak{q}_1(1_X)^{n-i-1} |0\rangle. \quad (3.21)$$

Combining (3.21) and (3.8), we conclude that

$$B_0(\gamma, n) = \frac{1}{(n-1)!} \cdot \mathfrak{q}_1(\gamma) \mathfrak{q}_1(1_X)^{n-1} |0\rangle = G_0(\gamma, n).$$

Next, we apply Lemma 2.17 to $\mathfrak{g} = \mathfrak{G}_1(\gamma)$, $A = \mathfrak{q}_1(1_X)^n |0\rangle$, $b = n$, and $a = 2$. So $\mathfrak{G}_1(\gamma)(A)$ consists of two parts (2.18) and (2.19). By Theorem 2.16 (v),

$$[\mathfrak{G}_1(\gamma), \mathfrak{q}_1(1_X)] = \mathfrak{q}'_1(\gamma) = \mathfrak{L}_1(\gamma).$$

Since $\mathfrak{G}_1(\gamma)|0\rangle = 0$ and $\mathfrak{L}_1(\gamma)|0\rangle = 0$, (2.18) is zero. By Theorem 2.16 (ii), $[[\mathfrak{G}_1(\gamma), \mathfrak{q}_1(1_X)], \mathfrak{q}_1(1_X)] = [\mathfrak{L}_1(\gamma), \mathfrak{q}_1(1_X)] = -\mathfrak{q}_2(\gamma)$. So (2.19) is equal to

$$-\binom{n}{2} \cdot \mathfrak{q}_2(\gamma) \mathfrak{q}_1(1_X)^{n-2}|0\rangle = -\frac{n!}{2} \cdot B_1(\gamma, n)$$

where we have used (3.21). Therefore, we obtain $\mathfrak{G}_1(\gamma)(A) = -n!/2 \cdot B_1(\gamma, n)$. Since $\mathfrak{G}_1(\gamma)(A) = G_1(\gamma, n) \cdot A = n! \cdot G_1(\gamma, n)$, we see that $B_1(\gamma, n) = -2G_1(\gamma, n)$.

(ii) In the proof of Lemma 3.5, we take $\alpha_1 = \gamma$, $i_0 = i$, $a = n_1 = i + 1$, $k = 1$, $A = \mathfrak{q}_1(1_X)^n|0\rangle$, and $b = n$. We see from (3.18) that

$$\mathfrak{G}_i(\gamma)(A) \equiv \binom{n}{i+1} \cdot (-1)^i \cdot \mathfrak{q}_1(1_X)^{n-i-1} \mathfrak{q}_{i+1}(\gamma)|0\rangle \pmod{\mathbb{H}'_{n,i}}. \quad (3.22)$$

Since $n! \cdot 1_{X^{[n]}} = A$, we conclude from (3.22) and (3.21) that

$$\begin{aligned} n! \cdot G_i(\gamma, n) &= G_i(\gamma, n) \cdot A = \mathfrak{G}_i(\gamma)(A) \\ &\equiv \binom{n}{i+1} \cdot (-1)^i \cdot \mathfrak{q}_1(1_X)^{n-i-1} \mathfrak{q}_{i+1}(\gamma)|0\rangle \pmod{\mathbb{H}'_{n,i}} \\ &\equiv \frac{n!}{(i+1)!} \cdot (-1)^i \cdot B_i(\gamma, n) \pmod{\mathbb{H}'_{n,i}}. \end{aligned}$$

It follows that $B_i(\gamma, n) \equiv (-1)^i (i+1)! \cdot G_i(\gamma, n) \pmod{\mathbb{H}'_{n,i}}$. \square

Theorem 3.23. *For $n \geq 1$, the cohomology ring $\mathbb{H}_n = H^*(X^{[n]})$ is generated by*

$$B_i(\gamma, n) = \mathfrak{B}_i(\gamma)(1_{X^{[n]}}) \quad (3.24)$$

where $0 \leq i < n$ and γ runs over a linear basis of $H^*(X)$. Moreover, the relations among these generators are precisely the relations among the restrictions $\mathfrak{B}_i(\gamma)|_{\mathbb{H}_n}$ of the corresponding operators $\mathfrak{B}_i(\gamma)$ to \mathbb{H}_n .

Proof. Note that the second statement follows from the fact that the operators $\mathfrak{B}_i(\gamma)|_{\mathbb{H}_n}$ are defined in terms of the cup products by the cohomology classes $B_i(\gamma, n)$. In the following, we prove the first statement.

Let $\widetilde{\mathbb{H}}_n$ be the subring generated by the $(n \cdot \dim H^*(X))$ cohomology classes in (3.24). In view of Theorem 3.19, it suffices to show that

$$G_i(\gamma, n) \in \widetilde{\mathbb{H}}_n \quad (3.25)$$

for all i with $0 \leq i < n$ and all $\gamma \in H^*(X)$. We shall use induction on i . First of all, this is true for $i = 0, 1$ by Lemma 3.20 (i). Next, fixing an integer i_0 with $2 \leq i_0 < n$, we assume that (3.25) is true for all the integers i with $0 \leq i < i_0$. We want to show that (3.25) holds as well for $i = i_0$, i.e.,

$$G_{i_0}(\gamma, n) \in \widetilde{\mathbb{H}}_n$$

for all $\gamma \in H^*(X)$. Indeed, we see from Lemma 3.20 (ii) that

$$G_{i_0}(\gamma, n) \equiv \frac{(-1)^{i_0}}{(i_0+1)!} \cdot B_{i_0}(\gamma, n) \pmod{\mathbb{H}'_{n,i_0}}. \quad (3.26)$$

By the definition of \mathbb{H}'_{n,i_0} in Lemma 3.5 and by the induction hypothesis, we have $\mathbb{H}'_{n,i_0} \subset \widetilde{\mathbb{H}}_n$. So it follows from (3.26) that $G_{i_0}(\gamma, n) \in \widetilde{\mathbb{H}}_n$. \square

We stress that Theorem 3.23 is not a mere consequence of Theorem 3.19 which was first established in [LQW]. As we worked on the only way that we found to derive Theorem 3.23, we happened to obtain a new proof of the old Theorem 3.19 which we present here.

In the last part of the paper, we shall establish the *transfer property* for the commutator $[\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)]$, cf. Definition 2.3 (iv). This property tells us how for a fixed $i \geq 0$, the operators $\mathfrak{B}_i(\gamma)$ associated with different classes $\gamma \in H^*(X)$ are related to each other. The transfer property, which seems to be universal for these types of operators in Hilbert schemes, was formulated and emphasized in [LQW], and examples of such property first appeared in [Leh]. Such a property can be used as a tool to show that various statements in Hilbert schemes are insensitive to the underlying surface X and can in principle be reduced to the understanding of the affine plane case. We mention that the transfer property for the commutator $[\mathfrak{G}_i(\gamma), \mathfrak{q}_n(\alpha)]$ established in [LQW] has been used effectively in a new remarkable paper of Lehn and Sorger [LS2]. We expect that the transfer property for $[\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)]$ which we prove here will play an important role in a further development.

We start with some notation, and prove a technical lemma which gives an alternative description of the cohomology class $B_i(\gamma, n)$. For $0 \leq i < n$, define

$$Z_{n,i+1} = \{(\xi, x) \in X^{[n]} \times X \mid \ell(\xi_x) \geq i+1\} \quad (3.27)$$

where ξ_x is the component of ξ such that ξ_x is supported at x . Note that $Z_{n,i+1}$ is closed, irreducible, and of dimension $(2n-i)$.

Lemma 3.28. *For $0 \leq i < n$ and $\gamma \in H^*(X)$, we have*

$$B_i(\gamma, n) = p_{1*}([\mathcal{Z}_{n,i+1}] \cdot p_2^* \gamma)$$

where p_1 and p_2 are the projections of $X^{[n]} \times X$ to $X^{[n]}$ and X respectively.

Proof. First of all, let $i = 0$ and $\gamma = 1_X$. By (3.2), we have

$$B_0(1_X, n) = \frac{1}{(n-1)!} \cdot \mathfrak{q}_1(1_X)^n |0\rangle = n \cdot 1_{X^{[n]}} = p_{1*}([\mathcal{Z}_n]) = p_{1*}([\mathcal{Z}_{n,1}])$$

noting that $1_{X^{[n]}} = 1/n! \cdot \mathfrak{q}_1(1_X)^n |0\rangle$. So Lemma 3.28 holds for $i = 0$ and $\gamma = 1_X$. In fact, by linearity, Lemma 3.28 is true for $i = 0$ and $\gamma \in H^0(X)$.

Next, we assume that either $i > 0$ or $\gamma \in H^s(X)$ with $s > 0$. Under these conditions, the Poincaré duals of the cohomology classes $B_i(\gamma, n)$ and

$$p_{1*}([\mathcal{Z}_{n,i+1}] \cdot p_2^* \gamma)$$

have clear geometric interpretations. So we shall work in the homology setting (only in this proof) with the help of Poincaré duality. Notice that via Poincaré duality, the cup product on cohomology theory become the intersection in homology theory, and the map p_{1*} we defined in the Conventions for cohomology theory becomes the ordinary pushforward map p_{1*} for homology theory. Let Γ be a $(4-s)$ -dimensional homology cycle of X representing the Poincaré dual of $\gamma \in H^s(X)$, and $X_0^{[m]}$ be the open dense subset of $X^{[m]}$ consisting of all $\xi \in X^{[m]}$ satisfying $|\text{Supp}(\xi)| = m$.

Now, the intersection of $\mathcal{Z}_{n,i+1}$ and $p_2^{-1}(\Gamma)$ contains an open dense subset U_0 which consists of all the points $(\xi, x) \in X^{[n]} \times X$ satisfying the conditions:

$$\ell(\xi_x) = i + 1, \xi - \xi_x \in X_0^{[n-i-1]}, \text{Supp}(\xi) \cap \Gamma = \{x\} \text{ if } s > 0.$$

Furthermore, the intersection $\mathcal{Z}_{n,i+1} \cap p_2^{-1}(\Gamma)$ is transversal along U_0 . Since either $i > 0$ or $s > 0$, the restriction $p_1|_{U_0}$ maps U_0 homeomorphically to the subset V_0 of $X^{[n]}$ consisting of all the points $\xi \in X^{[n]}$ satisfying the conditions:

$$\exists x \in \Gamma \text{ with } \ell(\xi_x) = i + 1, \xi - \xi_x \in X_0^{[n-i-1]}, \text{Supp}(\xi) \cap \Gamma = \{x\} \text{ if } s > 0.$$

It follows that the Poincaré dual of $p_{1*}([\mathcal{Z}_{n,i+1}] \cdot p_2^*\gamma)$ is the homology class represented by the closure of V_0 , denoted by $\overline{V_0}$:

$$\overline{V_0} = \{ \xi \in X^{[n]} \mid \exists x \in \Gamma \text{ with } \ell(\xi_x) \geq i + 1 \}.$$

Similarly, using (2.10) and induction, we conclude that the Poincaré dual of

$$B_i(\gamma, n) = \frac{1}{(n-i-1)!} \cdot \mathfrak{q}_{i+1}(\gamma) \mathfrak{q}_1(1_X)^{n-i-1} |0\rangle$$

is also represented by $\overline{V_0}$ (see [Na2]). So Lemma 3.28 follows. \square

Proposition 3.29. *Let $i \geq 0$, $n \in \mathbb{Z}$, and $\gamma, \alpha \in H^*(X)$. Then the commutator $[\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)]$ satisfies the transfer property, i.e., we have*

$$[\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)] = [\mathfrak{B}_i(1_X), \mathfrak{q}_n(\gamma\alpha)] = [\mathfrak{B}_i(\gamma\alpha), \mathfrak{q}_n(1_X)].$$

Proof. First of all, we notice that it suffices to show that

$$[\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)] = [\mathfrak{B}_i(1_X), \mathfrak{q}_n(\gamma\alpha)]. \quad (3.30)$$

Next, recall that $\mathfrak{q}_0(\alpha) = 0$. Also, by (2.7), (2.11) and (3.4), we have

$$[\mathfrak{B}_i(\gamma), \mathfrak{q}_{-n}(\alpha)] = [\mathfrak{B}_i(\gamma)^\dagger, (-1)^n \cdot \mathfrak{q}_n(\alpha)^\dagger] = (-1)^{n+1} \cdot [\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)]^\dagger.$$

So we need only to prove (3.30) for $n > 0$. In the following, let $n > 0$.

Consider the action of $[\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)]$ on $a \in H^*(X^{[m]})$. By the definition of the operators $\mathfrak{B}_i(\gamma)$ and $\mathfrak{q}_n(\alpha)$, we see that $\mathfrak{B}_i(\gamma)\mathfrak{q}_n(\alpha)(a)$ is equal to

$$(p_{n+m,1})_*([\mathcal{Z}_{n+m,i+1}] \cdot (p_{n+m,2})^*\gamma) \cdot \tilde{p}_{1*}([Q^{[m+n,m]}] \cdot \tilde{\rho}^*\alpha \cdot \tilde{p}_2^*a)$$

where $p_{n+m,1}, p_{n+m,2}$ are the projections of $X^{[m+n]} \times X$ to $X^{[m+n]}, X$ respectively, and $\tilde{p}_1, \tilde{\rho}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively. Using the projection formula and pulling all the cohomology classes $[\mathcal{Z}_{n+m,i+1}], (p_{n+m,2})^*\gamma, [Q^{[m+n,m]}], \tilde{\rho}^*\alpha, \tilde{p}_2^*a$ to $X^{[n+m],[1],[1],[m]}$, we conclude that

$$\mathfrak{B}_i(\gamma)\mathfrak{q}_n(\alpha)(a) = p_{1*}(p_{12}^*[\mathcal{Z}_{n+m,i+1}] \cdot p_{134}^*[Q^{[m+n,m]}] \cdot p_2^*\gamma \cdot p_3^*\alpha \cdot p_4^*a) \quad (3.31)$$

where for $1 \leq i_1 < \dots < i_s \leq 4$, the map $p_{i_1 \dots i_s}$ stands for the projection of $X^{[n+m], [1], [1], [m]}$ to the product of the i_1 -th, \dots , i_s -th factors. Similarly,

$$\mathfrak{q}_n(\alpha) \mathfrak{B}_i(\gamma)(a) = p_{1*}(p_{24}^*[\mathcal{Z}_{m,i+1}] \cdot p_{134}^*[Q^{[m+n,m]}] \cdot p_2^* \gamma \cdot p_3^* \alpha \cdot p_4^* a). \quad (3.32)$$

Claim. Regard $p_{12}^*[\mathcal{Z}_{n+m,i+1}] \cdot p_{134}^*[Q^{[m+n,m]}]$ and $p_{24}^*[\mathcal{Z}_{m,i+1}] \cdot p_{134}^*[Q^{[m+n,m]}]$ to be the products of algebraic cycles in the Chow ring $A^*(X^{[n+m], [1], [1], [m]})$. Then,

$$(p_{12}^*[\mathcal{Z}_{n+m,i+1}] - p_{24}^*[\mathcal{Z}_{m,i+1}]) \cdot p_{134}^*[Q^{[m+n,m]}] \in j_*(A^*(X^{[n+m]} \times \Delta_X \times X^{[m]})) \subset A^*(X^{[n+m], [1], [1], [m]}) \quad (3.33)$$

where Δ_X stands for the diagonal in $X^2 = X \times X$, and j is the inclusion

$$X^{[n+m]} \times \Delta_X \times X^{[m]} \hookrightarrow X^{[n+m], [1], [1], [m]}.$$

Proof. Denote a point in $X^{[n+m], [1], [1], [m]}$ by (ξ, x, y, η) . Let $U \subset X^{[n+m], [1], [1], [m]}$ be the open subset consisting of all the points (ξ, x, y, η) with $x \neq y$. Then the complement of U is precisely $X^{[n+m]} \times \Delta_X \times X^{[m]}$.

Consider $p_{12}^{-1} \mathcal{Z}_{n+m,i+1} \cap p_{134}^{-1} Q^{[m+n,m]}$ which has the expected dimension

$$2m + n + 1 - i. \quad (3.34)$$

A point $(\xi, x, y, \eta) \in U \cap (p_{12}^{-1} \mathcal{Z}_{n+m,i+1} \cap p_{134}^{-1} Q^{[m+n,m]})$ if and only if $x \neq y$, $\xi_x = \eta_x$ has length greater than or equal to $(i+1)$, and I_η/I_ξ has length n and support $\{y\}$, i.e., if and only if

$$\xi = \eta_x + \xi_y + \zeta \quad \text{and} \quad \eta = \eta_x + \eta_y + \zeta \quad (3.35)$$

where $x \neq y$, $\ell(\eta_x) \geq (i+1)$, $\eta_y \subset \xi_y$, $\ell(\xi_y) = n + \ell(\eta_y)$, $\{x, y\} \cap \text{Supp}(\zeta) = \emptyset$, and η_x, η_y, ξ_y are supported at x, y, y respectively. If $\eta_y \neq \emptyset$, then the dimension of the set of those points (ξ, x, y, η) satisfying (3.35) is at most

$$\begin{aligned} & \#(\text{moduli of } x, y) + (\ell(\eta_x) - 1) + (\ell(\eta_y) - 1) + (\ell(\xi_y) - 1) + 2\ell(\zeta) \\ &= 2m + n + 1 - \ell(\eta_x) < 2m + n + 1 - i. \end{aligned} \quad (3.36)$$

If $\eta_y = \emptyset$, then the dimension of the set of the points (ξ, x, y, η) satisfying (3.35) is

$$\begin{aligned} & \#(\text{moduli of } x, y) + \#(\text{moduli of } \eta_x) + (\ell(\xi_y) - 1) + 2\ell(\zeta) \\ &= 2m + n + 1 - \#(\text{moduli of } \eta_x) = 2m + n + 1 - i. \end{aligned} \quad (3.37)$$

By (3.34), (3.36) and (3.37), $U \cap (p_{12}^{-1} \mathcal{Z}_{n+m,i+1} \cap p_{134}^{-1} Q^{[m+n,m]})$ contains the open dense subset $V \stackrel{\text{def}}{=} \{(\xi, x, y, \eta) \mid (\xi, x, y, \eta) \text{ satisfies (3.35) with } \eta_y = \emptyset\}$ which is also irreducible. Now since $x \neq y$, the intersection $p_{12}^{-1} \mathcal{Z}_{n+m,i+1} \cap p_{134}^{-1} Q^{[m+n,m]}$ along V is transversal. So using the refined intersection [Ful], we conclude that

$$p_{12}^*[\mathcal{Z}_{n+m,i+1}] \cdot p_{134}^*[Q^{[m+n,m]}] = [\overline{V}] + j_*(b_1) \quad (3.38)$$

where \overline{V} is the closure of V in $X^{[n+m], [1], [1], [m]}$ and $b_1 \in A^*(X^{[n+m]} \times \Delta_X \times X^{[m]})$.

Similarly, we see that $p_{24}^*[\mathcal{Z}_{m,i+1}] \cdot p_{134}^*[Q^{[m+n,m]}] = [\overline{V}] + j_*(b_2)$ for some algebraic cycle $b_2 \in A^*(X^{[n+m]} \times \Delta_X \times X^{[m]})$. Combining this with (3.38) yields (3.33). \square

Now we continue the proof of (3.30). Recalling our conventions established at the end of section 1, we see from (3.31), (3.32) and (3.33) that

$$\begin{aligned}
& [\mathfrak{B}_i(\gamma), \mathfrak{q}_n(\alpha)](a) \\
&= p_{1*}((p_{12}^*[\mathcal{Z}_{n+m, i+1}] - p_{24}^*[\mathcal{Z}_{m, i+1}]) \cdot p_{134}^*[Q^{[m+n, m]}] \cdot p_2^*\gamma \cdot p_3^*\alpha \cdot p_4^*a) \\
&= \bar{p}_{1*}(b \cdot \bar{p}^*(\gamma\alpha) \cdot \bar{p}_2^*a)
\end{aligned} \tag{3.39}$$

where $b \in H^*(X^{[n+m]} \times \Delta_X \times X^{[m]})$ is independent of γ, α and a , and $\bar{p}_1, \bar{p}, \bar{p}_2$ are the projections of $X^{[m+n]} \times \Delta_X \times X^{[m]}$ to $X^{[m+n]}, \Delta_X \cong X, X^{[m]}$ respectively. Since b is independent of γ, α and a , (3.30) follows immediately from (3.39). \square

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